②

NASA Contractor Report 181890

ICASE Report No. 89-16

AD-A212 313

# ICASE

ESTIMATION OF NONLINEAR DAMPING
IN SECOND ORDER DISTRIBUTED
PARAMETER SYSTEMS

H. T. Banks
Simeon Reich
I. G. Rosen

**NASA**

National Aeronautics and
Space Administration

**Langley Research Center**
Hampton, Virginia 23665-5225

**89   9   14   010**

# ESTIMATION OF NONLINEAR DAMPING IN SECOND ORDER DISTRIBUTED PARAMETER SYSTEMS[1]

## H.T. Banks [2]

Center for Control Sciences
Division of Applied Mathematics
Brown University
Providence, RI 02912, U.S.A.

Center for Applied Mathematical Sciences
Department of Mathematics
University of Southern California
Los Angeles, CA 90089, U.S.A.

## Simeon Reich [3]

Center for Applied Math. Sciences
Department of Mathematics
University of Southern California
Los Angeles, CA 90089, U.S.A.

Inst. of Advanced Studies in Mathematics
Department of Mathematics
The Technion-Israel Institute of Technology
Haifa 32000, Israel

## I.G. Rosen [4]

Center for Applied Mathematical Sciences
Department of Mathematics
University of Southern California
Los Angeles, CA 90089, U.S.A.

Abstract. An approximation and convergence theory for the identification of nonlinear damping in abstract wave equations is developed. It is assumed that the unknown dissipation mechanism to be identified can be described by a maximal monotone operator acting on the generalized velocity. The stiffness is assumed to be linear and symmetric. Functional analytic techniques are used to establish that solutions to a sequence of finite dimensional (Galerkin) approximating identification problems in some sense approximate a solution to the original infinite dimensional inverse problem.

i

# 1. Introduction

In recent investigations [2], [3], [0], Banks, Inman, and their colleagues carried out a series of experimental and computational investigations on linear damping models for composite material flexible structures. Other investigations [1], [7] involving experiments with flexible structures point to the need to understand nonlinear aspects (in particular, damping) of flexible structures. In this paper, we present a first step toward development of a rigorous theoretical foundation for computational methods to study nonlinear mechanisms in such structures.

Traditionally, nonlinearities in structural models usually arise when one needs to consider large amplitude displacements. However a rather substantial engineering literature (see Chap. 7 of [19]) exists on a wide variety of "nonlinear" effects in structures. There are numerous sources proposed for these nonlinearities; they can be geometric (nonlinear stretching; large curvature) or material (nonlinear stress-strain laws; nonlinear damping) in nature. In mathematical models these can be manifested in nonlinear boundary conditions, nonlinear stiffness operators, and/or nonlinear damping operators. For example, in the usual derivations [11], [20], [24] for the Euler-Bernoulli theory of beams or the Love-Kirchoff theory of plates, one must make several "linearizing" assumptions. The commonly postulated Hooke's law is a linear stress-strain constitutive law. Even under this assumption, one has a nonlinear moment-curvature relationship (and hence a nonlinear stiffness operator in the resulting partial differential equation model) unless one makes the usual approximations (e.g., see Chap. 11 of [20]) in the curvature expression $1/R = u_{xx}/\left[1 + u_x^2\right]^{3/2}$ arising in elementary beam theories.

While the ideas presented below can be used to develop a methodology to study nonlinear stiffness operators (which, of course, can be of great importance in motions involving large geometries or nonlinear material properties), we shall focus here on a theory for nonlinear damping operators. Our approach will allow one to study damping mechanisms that do not decouple across the modes of the structure and this has substantial practical significance.

1

The efforts of Balakrishnan, Taylor, and their colleagues (see [1] and the discussions and references therein) have produced evidence that in complex structures such as the SCOLE structure at NASA Langley, one cannot use linear damping models where the damping decouples across the "linear undamped modes" to successfully describe experimental observations. The presentation in [1] suggests that modal-based nonlinear damping might be adequate. However, in light of the results in [3], [9], where it is found that for even simple structures one can be in the unfortunate situation where damping does not decouple across the modes, this issue is far from settled. Indeed, it is not clear whether such damping should be modeled with a nonlinearity or whether it might be adequately modeled with a linear mechanism that does not decouple (i.e. cannot be described via modal damping ratios in the usual engineering manner). Whatever the case, it is clear that further investigations are necessary. Thus one requires methods to study linear and nonlinear damping mechanisms in models of second order systems where one does not assume that the damping model can be simplified by modal representations.

Our goal then is to develop such a methodology for nonlinear systems. The theoretical basis for the linear studies in [2], [3], [9] was an abstract approximation framework developed by Banks and Ito in [4]. It is our purpose here to develop a theory for nonlinear second order systems that parallels that of [4]. Since we shall use monotone operator theory and nonlinear evolution systems (a theory which itself has been shown to possess substantial limitations in treating very general nonlinearities), we expect this to be only an initial contribution from our studies. However, the theory presented below will allow us to treat a large class of nonlinear damping operators that are thought to be of practical importance.

We develop an abstract approximation framework and convergence theory for the identification of nonlinear dissipation or damping mechanisms in second order distributed parameter systems. More precisely, we consider the estimation of nonlinear maximal monotone damping operators in abstract wave equations with linear symmetric stiffness operators. Our treatment here is in the spirit of our earlier work on first order systems in [8] and

is an extension of Banks' and Ito's results in [4] concerning the identification of second order linear systems. By rewriting the underlying second order initial value problem as an equivalent first order system in an appropriate product space, we can, in principle, apply the abstract approximation theory developed in [8]. However, since the resulting first order dynamics are not in general described by a strongly monotone operator, the results we present below cannot be obtained via a direct application of our first order theory. Only the essential ideas underlying the general abstract Banach space theory involving m-accretive operators given in Section 2 of [8] are directly applicable.

In Section 2 below we review the abstract existence, uniqueness, regularity, and approximation results for nonlinear evolution equations in Banach space with dynamics governed by m-accretive operators. In the third section we consider inverse problems for abstract wave equations with nonlinear damping. We reformulate these second order equations as equivalent first order systems, and discuss existence and regularity of solutions. The approximation theory and convergence results are presented in Section 4. In the concluding section we illustrate the application of our theory with an example involving estimation of nonlinear velocity dependent damping in a one dimensional wave equation.

## 2. Abstract Nonlinear Evolution Equations in Banach Spaces

Let $X_0$ be a Banach space with norm $|\cdot|_0$. Let $T > 0$ and suppose that $F_0 \in L_1(0, T; X_0)$. Let $A_0 : X_0 \to 2^{X_0}$ be an, in general, nonlinear set-valued operator with $\text{Dom}(A_0) = \{x \in X_0 : A_0 x \neq \emptyset\}$. We assume that for some $\omega \in \mathbf{R}$ the operator $A_0 + \omega I$ is m-accretive. That is, that (i) $|x_1 - x_2|_0 \leq |(1 + \lambda\omega)(x_1 - x_2) + \lambda(y_1 - y_2)|_0$ for every $x_1, x_2 \in \text{Dom}(A_0), y_1 \in A_0 x_1, y_2 \in A_0 x_2$, and $\lambda > 0$, and (ii) $\mathcal{R}(I + \lambda(A_0 + \omega I)) \equiv \bigcup_{x \in \text{Dom}(A_0)} (I + \lambda(A_0 + \omega I)) x = X_0$ for some $\lambda > 0$. It is well known (see [10]) that the operator $A_0 + \omega I$ being m-accretive implies that for each $\lambda > 0$ the resolvent of $A_0 + \omega I$ at $\lambda, J(\lambda; A_0 + \omega I) : X \to X$, given by $J(\lambda; A_0 + \omega I) = (I + \lambda(A_0 + \omega I))^{-1}$ is a single valued, everywhere defined, nonexpansive (nonlinear) operator on $X_0$.

3

We consider the quasi-autonomous initial value problem

$$(2.1) \qquad \dot{x}(t) + A_0 x(t) \ni F_0(t), \quad 0 < t \le T$$

$$(2.2) \qquad x(0) = x^0$$

with $x^0 \in \text{Dom}(A_0)$. By a (strong) solution to the initial value problem (2.1), (2.2) we shall mean a strongly continuous function $x : [0, T] \to X_0$ which is absolutely continuous on compact subintervals of $(0, T)$, differentiable almost everywhere on $(0, T)$, and satisfies $F_0(t) - \dot{x}(t) \in A_0 x(t)$ for almost every $t \in (0, T)$ and $x(0) = x^0$.

A two parameter family of nonlinear operators, $\{U_0(t, s) : 0 \le s \le t \le T\}$, defined on a subset $\Omega$ of $X_0$ is said to be a nonlinear evolution system on $\Omega$ if it satisfies (i) $U_0(t, s)U_0(s, r)\varphi = U_0(t, r)\varphi$ for every $\varphi \in \Omega$ and $r, s, t$ with $0 \le r \le s \le t \le T$, and (ii) the mapping $(t, s) \to U_0(t, s)\varphi$ is continuous from the triangle $\Delta = \{(\tau, \sigma) : 0 \le \sigma \le \tau \le T\} \subset \mathbf{R}^2$ into $X_0$ for each $\varphi \in \Omega$.

It can be shown (see [8], [12], and [14]) that with $A_0, F_0$, and $x_0$ as they have been defined above, there exists a unique nonlinear evolution system $\{U_0(t, s) : 0 \le s \le t \le T\}$ on $\overline{\text{Dom}(A_0)}$ which satisfies

(i) $|U_0(t, s)\varphi - U_0(t, s)\psi|_0 \le e^{\omega(t-s)} |\varphi - \psi|_0$, for all $\varphi, \psi \in \overline{\text{Dom}(A_0)}$ and $(t, s) \in \Delta$,

(ii) $|U_0(s + t, s)\varphi - U_0(r + t, r)\varphi|_0 \le 2 \int_0^t e^{\omega(t-\tau)} |F_0(\tau + s) - F_0(\tau + r)|_0 \, d\tau$, for all $\varphi \in \overline{\text{Dom}(A_0)}$ and all $t > 0$ such that $s + t, r + t \le T$,

(iii) If the initial value problem (2.1), (2.2) has a strong solution $x$, then $x(t) = U_0(t, s)x(s)$ for all $(t, s) \in \Delta$.

The strongly continuous function $x$ defined by $x(t) = U_0(t, 0)x^0, t \in [0, T]$, is sometimes referred to as the unique mild or generalized solution to (2.1), (2.2). It is immediately clear from (iii) above that when the initial value problem (2.1), (2.2) has a strong solution, it coincides with the unique mild solution defined in terms of the nonlinear evolution

4

system $\{U_0(t,s) : 0 \leq s \leq t \leq T\}$ on $\overline{\text{Dom}(A_0)}$. Theorem 2.1 below is the fundamental abstract approximation result upon which the approximation and convergence theory for second order systems to be developed below will be based. It is similar in spirit to the approximation results for nonlinear evolution systems given in [13] and [15].

For Banach spaces $X$ and $Y$ we denote the Banach space of continuous linear operators from $X$ into $Y$ by $\mathcal{L}(X,Y)$.

**Theorem 2.1** For each $n \in \mathbf{Z}^+ = \{1,2,3,\dots\}$, let $X_n$ be a Banach space with norm $|\cdot|_n$. Let $\pi_n \in \mathcal{L}(X_0, X_n)$ with $|\pi_n \varphi|_n \leq \nu |\varphi|_0$ where $\nu$ is a positive constant independent of $n$. Let $F_n \in L_1(0,T;X_n)$ and let $A_n : X_n \to 2^{X_n}$ be an operator on $\text{Dom}(A_n) \subset X_n$ with $A_n + \omega I$ m-accretive. Suppose that there exists a function $g \in L_1(0,T)$ for which $|F_n(t)|_n \leq g(t)$, for almost every $t \in (0,T)$ and that

(i) for each $\varphi \in \overline{\text{Dom}(A_0)}$ there exists $\{\varphi_n\}_{n=1}^\infty$ with $\varphi_n \in \overline{\text{Dom}(A_n)}$ such that
$$\lim_{n \to \infty} |\varphi_n - \pi_n \varphi|_n = 0,$$

(ii) $\lim_{n \to \infty} |F_n(t) - \pi_n F_0(t)|_n = 0$, a.e. $t \in (0,T)$,

(iii) for some $\lambda > 0$, we have $\lim_{n \to \infty} |J(\lambda; A_n + \omega I)\varphi_n - \pi_n J(\lambda; A_0 + \omega I)\varphi|_n = 0$ whenever $\varphi_n \in X_n$ and $\varphi \in X_0$ with $\lim_{n \to \infty} |\varphi_n - \pi_n \varphi|_n = 0$. Then if $\{U_n(t,s) : 0 \leq s \leq t \leq T\}$ is the evolution system on $\overline{\text{Dom}(A_n)}$ generated by $A_n$ and $F_n$, we have

(2.3) $$\lim_{n \to \infty} |U_n(t,s)\varphi_n - \pi_n U_0(t,s)\varphi|_n = 0$$

uniformly in $s$ and $t$ for $(t,s) \in \Delta$, whenever $\varphi \in \overline{\text{Dom}(A_0)}$ and $\varphi_n \in \overline{\text{Dom}(A_n)}$, with $\lim_{n \to \infty} |\varphi_n - \pi_n \varphi|_n = 0$.

The proof of Theorem 2.1, which we omit, is essentially the same as the proof of Theorem 2.2 in [8]. Some modification, albeit relatively simple and straightforward, is necessary however due to the fact that in our application of Theorem 2.1 below we cannot choose the $X_n$'s as subspaces of $X_0$. In the proof this is handled by simply expressing all convergence conditions and results in terms of the mappings $\pi_n$. (Compare the statements of Theorem

2.1 above and Theorem 2.2 in [8]). We note that the convergence result (2.3) is closely related to the notion of factor convergence as given in [16]. We remark that this situation $(X_n \not\subset X_0)$ arises frequently in parameter estimation problems and the reader can find a more complete discussion with examples along with a linear semigroup version of the above approximation theorem in [5, Chapter II]. Whether or not (2.3) implies that $U_n(t,s)\varphi_n$ and $U_0(t,s)\varphi$ are close in any reasonable or useful sense depends of course upon how the spaces $X_n$, the norms $|\cdot|_n$, and the mappings $\pi_n$ are chosen. In our treatment below we shall be able to apply the following corollary, which follows from Theorem 2.1 via a straightforward application of the triangle inequality.

<u>Corollary 2.1</u>   Suppose that there exists a Banach space $X$ with norm $|\cdot|_X$ for which $X_0$ is set-equivalent to $X$, for each $n = 1, 2, \ldots, X_n$ is a subspace of $X$, and all of the norms $|\cdot|_n, n = 0, 1, 2, \ldots$ are uniformly equivalent to $|\cdot|_X$. Suppose further that $\lim_{n\to\infty} \pi_n\varphi = \varphi, \varphi \in X$. Then conditions (i) - (iii) in the statement of Theorem 2.1 can be replaced by

(i) $\overline{\mathrm{Dom}(A_0)} \subset \underset{n\to\infty}{Li} \overline{\mathrm{Dom}(A_n)}$ , where $Li$ denotes the lower limit of the sequence of sets, $\{\overline{\mathrm{Dom}(A_n)}\}$ (see [11], p. 335),

(ii) $\lim_{n\to\infty} F_n(t) = F_0(t)$ in $X$ for a.e. $t \in (0,T)$, and

(iii) for some $\lambda > 0$, $\lim_{n\to\infty} J(\lambda; A_n + \omega I)\varphi_n = J(\lambda; A_n + \omega I)\varphi$ in $X$ for each $\varphi \in X$ whenever $\varphi_n \in X_n$ with $\lim_{n\to\infty} \varphi_n = \varphi$ in $X$. Moreover, the conclusion (2.3) can be replaced by

$$\lim_{n\to\infty} U_n(t,s)\varphi_n = U_0(t,s)\varphi$$

in $X$, uniformly in $t$ and $s$ for $(t,s) \in \Delta$, for each $\varphi \in \overline{\mathrm{Dom}(A_0)}$ and $\varphi_n \in \overline{\mathrm{Dom}(A_n)}, n = 1, 2, \ldots$ with $\lim_{n\to\infty} \varphi_n = \varphi$ in $X$.

## 3. The Identification of Second Order Systems

Although the identification of nonlinear dissipation mechanisms in infinite dimensional second order (mechanical) systems is of primary interest to us here, we treat the somewhat

more general problem which includes the simultaneous estimation of linear stiffness. It will become clear from our discussions below that an approach similar to the one we use here could be used to develop techniques for the estimation of nonlinear stiffness in the presence of linear damping. The extension of our general approach to handle inverse problems involving the simultaneous identification of both nonlinear stiffness and damping is at present still under investigation. We hope to be able to report on results in this direction in the not too distant future.

Let $H$ be a real Hilbert space with inner product $< \cdot, \cdot >$ and corresponding norm $|\cdot|$. Let $V$ be a reflexive Banach space with norm denoted by $\| \cdot \|$, and assume that $V$ is densely and continuously embedded in $H$. If we let $V^*$ denote the dual of $V$ it follows that $V \hookrightarrow H \hookrightarrow V^*$ with the embedding of $H$ in $V^*$ dense and continuous. The continuity of the embeddings implies the existence of a constant $\mu > 0$ for which $|\varphi| \leq \mu \|\varphi\|, \varphi \in V$, and $\|\varphi\|_* \leq \mu |\varphi|, \varphi \in H$, where $\|\cdot\|_*$ denotes the uniform operator norm on $V^*$. We shall also use $< \cdot, \cdot >$ to denote the usual extension of the $H$ inner product to the duality pairing between $V$ and $V^*$. Let $Q$ and $Z$ be metric spaces and let $Q$ be a fixed, nonempty, compact subset of $Q$.

For each $q \in Q$ let the operator $\mathcal{A}(q) \in \mathcal{L}(V, V^*)$ satisfy the following conditions:

(A1) (Symmetry) For all $\varphi, \psi \in V, < \mathcal{A}(q)\varphi, \psi > = < \varphi, \mathcal{A}(q)\psi >$;

(A2) (Continuity) For each $\varphi \in V$ the map $q \to \mathcal{A}(q)\varphi$ is continuous from $Q \subset Q$ into $V^*$;

(A3) (Equi-V-Coercivity) There exist constants $\alpha_1 \in \mathbf{R}$ and $\alpha_0 > 0$, both independent of $q \in Q$ for which $< \mathcal{A}(q)\varphi, \varphi > + \alpha_1 |\varphi|^2 \geq \alpha_0 \|\varphi\|^2$, for all $\varphi \in V$ and $q \in Q$;

(A4) (Equi-Boundedness) The operators $\mathcal{A}(q)$ are uniformly bounded in $q$ for $q \in Q$; that is, there exists a constant $\alpha > 0$, independent of $q \in Q$, for which $\|\mathcal{A}(q)\varphi\|_* \leq \alpha \|\varphi\|$, for all $\varphi \in V$.

7

In addition, for each $q \in Q$ let the operator $\mathcal{B}(q) : \mathrm{Dom}(\mathcal{B}(q)) \subset V \to 2^{V^*}$ satisfy the following conditions:

(B1) (Domain) $\mathrm{Dom}(\mathcal{B}(q)) = \mathrm{Dom}(\mathcal{B})$ is independent of $q$ for $q \in Q$, and $0 \in \mathrm{Dom}(\mathcal{B})$;

(B2) (Continuity) For each $\varphi \in \mathrm{Dom}(\mathcal{B})$, the map $q \to (\mathcal{B}(q)\varphi$ is lower semi-continuous from $Q \subset \mathcal{Q}$ into $2^{V^*}$ (see [17], Section 18.i, page 173);

(B3) (Maximal Monotonicity) For all $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{B}_q \equiv \{(\varphi, \psi) \in V \times V^* : \varphi \in \mathrm{Dom}(\mathcal{B}), \psi \in \mathcal{B}(q)\varphi\}$ we have $< \psi_1 - \psi_2, \varphi_1 - \varphi_2 > \ge 0$ with $\mathcal{B}_q$ not properly contained in any other subset of $V \times V^*$ for which this monotonicity condition holds;

(B4) (Equi-Boundedness) The operators $\mathcal{B}(q)$ map $V$-bounded subsets of $\mathrm{Dom}(\mathcal{B})$ into subsets of $V^*$ which are uniformly $V^*$-bounded in $q$ for $q \in Q$; that is, if $S$ is a $V$-bounded subset of $\mathrm{Dom}(\mathcal{B})$, then the set $\mathcal{B}(q)S$ is $V^*$-bounded, uniformly in $q$ for $q \in Q$.

For each $q \in Q$ let $u^0(q) \in V, u^1(q) \in H$ and $f(\cdot; q) \in L_1(0, T; H)$ where $T > 0$ is given and fixed. We shall require that the mappings $q \to u^0(q) \in V, q \to u^1(q) \in H$, and $q \to f(t; q)$, for almost every $t \in (0, T)$, are continuous from $Q \subset \mathcal{Q}$ into $V, H$, and $H$ respectively. For each $z \in Z$, let $\phi(\cdot \; ; z)$ be a continuous mapping from $C(0, T; V \times H)$ into $\mathbf{R}^+$ and consider the identification problem given by:

(ID) Given observations $z \in Z$, determine parameters $\bar{q} \in Q$ which minimize the performance index

$$\Phi(q) = \phi((u(q), \dot{u}(q)) ; z)$$

where for each $q \in Q, u(q) = u(\cdot \; ; q)$ is the mild solution to the second order initial value problem

$$(3.1) \qquad \ddot{u}(t) + \mathcal{B}(q)\dot{u}(t) + \mathcal{A}(q)u(t) \ni f(t; q), \qquad 0 < t < T$$

$$(3.2) \qquad u(0) = u^0(q), \qquad \dot{u}(0) = u^1(q).$$

8

Following Barbu [10], we rewrite the second order initial value problem (3.1), (3.2) as an equivalent first order system in a product space and then apply the abstract theory outlined in Section 2 above to make the notion of a mild solution precise. Let $X := V \times H$ be the Banach space endowed with the norm $| \cdot |$ given by

$$|(\varphi, \psi)|_X = \left( \|\varphi\|^2 + |\psi|^2 \right)^{\frac{1}{2}}.$$

In addition, for each $q \in Q$ let $\mathcal{H}(q)$ be the Hilbert space which is set equivalent to $X$ and has as inner product $< \cdot, \cdot >_q$ given by

(3.3) $\qquad < (\varphi_1, \psi_1), (\varphi_2, \psi_2) >_q = < \mathcal{A}(q)\varphi_1, \varphi_2 > + \alpha_1 < \varphi_1, \varphi_2 > + < \psi_1, \psi_2 >.$

Denote the corresponding induced norm by $| \cdot |_q$. Conditions (A1) - (A4) above guarantee that (3.3) indeed defines an inner product on $X$ and that the Banach spaces $\{\mathcal{H}(q), |\cdot|_q\}$ are uniformly (in $q$, for $q \in Q$) norm equivalent to the space $\{X, | \cdot |_X\}$.

For each $q \in Q$ define the operator $A(q) : \mathrm{Dom}(A(q)) \subset \mathcal{H}(q) \to 2^{\mathcal{H}(q)}$ by

$$A(q)(\varphi, \psi) = (-\psi, \{\mathcal{A}(q)\varphi + \mathcal{B}(q)\psi\} \cap H),$$

with $\mathrm{Dom}(A(q)) = \{(\varphi, \psi) \in V \times V : \psi \in \mathrm{Dom}(\mathcal{B}), \{\mathcal{A}(q)\varphi + \mathcal{B}(q)\psi\} \cap H \neq \emptyset\}$.

<u>Theorem 3.1</u>  There exists an $\omega \in \mathbf{R}$, independent of $q \in Q$, for which the operator $A(q) + \omega I$ is m-accretive.

<u>Proof.</u>  Our proof is analogous to the one given by Barbu [10] in the case where the space and operator do not depend upon a parameter. We first show that for $\omega \in \mathbf{R}$ sufficiently large the operator $A(q) + \omega I$ is monotone. For $(\varphi, \psi), (\eta, \theta) \in \mathrm{Dom}(A(q))$ and $\omega \in \mathbf{R}$ let $(\hat{\varphi}, \hat{\psi}) \in (A(q) + \omega I)(\varphi, \psi)$ and $(\hat{\eta}, \hat{\theta}) \in (A(q) + \omega I)(\eta, \theta)$. It follows that $\hat{\varphi} = -\psi + \omega\varphi, \hat{\eta} = -\theta + \omega\eta, \hat{\psi} = \mathcal{A}(q)\varphi + \tilde{\psi} + \omega\psi$ and $\hat{\theta} = \mathcal{A}(q)\eta + \tilde{\theta} + \omega\theta$ for some $\tilde{\psi} \in \mathcal{B}(q)\psi$ and some $\tilde{\theta} \in \mathcal{B}(q)\theta$. Then, using the monotonicity of the operator $\mathcal{B}(q)$

9

(condition (B3)) we obtain

$$< (\hat{\varphi}, \hat{\psi}) - (\hat{\eta}, \hat{\theta}), (\varphi, \psi) - (\eta, \theta) >_q$$

$$= < (-\psi + \theta, \mathcal{A}(q)(\varphi - \eta) + \tilde{\psi} - \tilde{\theta}), (\varphi - \eta, \psi - \theta) >_q + \omega \, |(\varphi - \eta, \psi - \theta)|_q^2$$

$$= -\alpha_1 < \psi - \theta, \varphi - \eta > + < \tilde{\psi} - \tilde{\theta}, \psi - \theta > + \omega \, |(\varphi - \eta, \psi - \theta)|_q^2$$

$$\geq -\alpha_1 < \psi - \theta, \varphi - \eta > + \omega < \mathcal{A}(q)(\varphi - \eta), \varphi - \eta > + \omega \alpha_1 \, |\varphi - \eta|^2 + \omega \, |\psi - \theta|^2$$

$$\geq 0$$

if

$$\omega \geq \omega_0 \equiv \sup \left\{ \frac{\alpha_1 < u, v >}{< A(q) u, u > + \alpha_1 \, |u|^2 + |v|^2} ; (u, v) \in X, (u, v) \neq 0, q \in Q \right\}.$$

We note that $\omega_0$ exists since

$$\frac{\alpha_1 < u, v >}{< A(q) u, v > + \alpha_1 \, |u|^2 + |v|^2} \leq \frac{\alpha_1 \mu \, \|u\| \, |v|}{\alpha_0 \, \|u\|^2 + |v|^2}$$

$$\leq \frac{\frac{1}{2} \alpha_1 \mu \left\{ \|u\|^2 + |v|^2 \right\}}{\alpha_0 \, \|u\|^2 + |v|^2} \leq \frac{\alpha_1 \mu}{2 \min (\alpha_0, 1)}.$$

If $\omega_0 < 0$, we set $\omega = 0$ and henceforth assume without loss of generality that $\omega \geq 0$. Since $A(q) + \omega I$ is monotone, it is also accretive. Indeed, letting $\lambda > 0$, we see that

$$|(\varphi, \psi) - (\eta, \theta)|_q^2 \leq |(\varphi, \psi) - (\eta, \theta)|_q^2$$

$$+ \lambda < (\hat{\varphi}, \hat{\psi}) - (\hat{\eta}, \hat{\theta}), (\varphi, \psi) - (\eta, \theta) >_q$$

$$= < (\varphi, \psi) - (\eta, \theta) + \lambda \{ (\hat{\varphi}, \hat{\psi}) - (\hat{\eta}, \hat{\theta}) \}, (\varphi, \psi) - (\eta, \theta) >_q$$

$$\leq |(\varphi, \psi) - (\eta, \theta) + \lambda \{ (\hat{\varphi}, \hat{\psi}) - (\hat{\eta}, \hat{\theta}) \} |_q \, |(\varphi, \psi) - (\eta, \theta)|_q.$$

Thus we obtain

$$|(\varphi, \psi) - (\eta, \theta)|_q \leq |(\varphi, \psi) - (\eta, \theta) + \lambda \{ (\hat{\varphi}, \hat{\psi}) - (\hat{\eta}, \hat{\theta}) \} |_q.$$

To demonstrate m-accretivity, we let $\lambda > 0$, and let $(\eta, \theta) \in \mathcal{H}(q)$ be fixed but arbitrary, and show that $(I + \lambda(A(q) + \omega I))(\varphi, \psi) \ni (\eta, \theta)$ for some $(\varphi, \psi) \in \text{Dom}(A(q))$. This statement can be equivalently written as

(3.4) $$\varphi + \lambda(-\psi + \omega \varphi) = \eta$$

$$(3.5) \qquad \psi + \lambda \left( \mathcal{A}(q) \varphi + \mathcal{B}(q)\psi + \omega \psi \right) \ni \theta.$$

Solving for $\varphi$ in (3.4) and the substituting into (3.5), we obtain

$$(3.6) \qquad \varphi = (1 + \lambda \omega)^{-1} (\lambda \psi + \eta)$$

$$(3.7) \qquad (1 + \lambda \omega) \psi + \lambda^2 (1 + \lambda \omega)^{-1} \mathcal{A}(q) \psi + \lambda \mathcal{B}(q) \psi \ni \theta - \lambda (1 + \lambda \omega)^{-1} \mathcal{A}(q) \eta.$$

Define the operator $T_\lambda(q) \in \mathcal{L}(V, V^*)$ by

$$(3.8) \qquad T_\lambda(q) = (1 + \lambda \omega) I + \lambda^2 (1 + \lambda \omega)^{-1} \mathcal{A}(q).$$

It follows from condition (A3) that for $u \in V$

$$(3.9) \qquad < T_\lambda(q) u, u > \; \geq \; \tau_1(\lambda) |u|^2 + \tau_0(\lambda) \|u\|^2,$$

where $\tau_1(\lambda) = (1 + \lambda \omega) - \lambda^2 (1 + \lambda \omega)^{-1} \alpha_1$ and $\tau_0(\lambda) = \lambda^2 (1 + \lambda \omega)^{-1} \alpha_0$. For $\lambda > 0$ appropriately chosen (small if $\alpha_1 > 0$; any $\lambda > 0$ if $\alpha_1 \leq 0$), it is seen that $\tau_1(\lambda) > 0$ and hence that $T_\lambda(q)$ is monotone and coercive in the sense of Barbu [10, p. 34]. That is,

$$< T_\lambda(q) u, u > / \|u\| \to \infty, \qquad \text{as} \quad \|u\| \to \infty.$$

Therefore, applying Barbu's [10] Corollary II.1.3 with his $X$, B and $A$ chosen as $X = V, B = T_\lambda(q)$ and $A = \lambda \mathcal{B}(q)$, we find that $\mathcal{R}(T_\lambda(q) + \lambda \mathcal{B}(q)) = V^*$. Consequently there exists a $\psi \in \text{Dom}(\mathcal{B}(q))$ for which (3.7) holds. If we then obtain $\varphi$ from (3.6), it follows that $\mathcal{R}(I + \lambda (A(q) + \omega I)) = \mathcal{H}(q)$ and the theorem is proved.

For each $q \in Q$ define $F \in L_1(0, T; \mathcal{H}(q))$ by $F(t; q) = (0, f(t; q))$, a.e. $t \in (0, T)$, and set $x^0(q) = \left( u^0(q), u^1(q) \right) \in \mathcal{H}(q)$. Theorem 3.1 and the discussions of Section 2 yield that $A(q)$ and $F(\cdot; q)$ generate a nonlinear evolution system $\{ U(t, s; q) : 0 \leq s \leq t \leq T \}$ on $\overline{\text{Dom}(A(q))}$. Henceforth we shall assume that $x^0(q) \in \overline{\text{Dom}(A(q))}$ for each $q \in Q$, and by a mild solution, $u(q)$, to the second order initial value problem (3.1), (3.2) we shall mean the $V$-continuous (recall the uniform norm equivalence of $X$ and $\mathcal{H}(q)$) function $u(\cdot; q)$ given by

the first component of the $\mathcal{H}(q)$ (or $X$) continuous function $x\left(t;q\right) = U\left(t,0;q\right)x^0\left(q\right), t \in [0,T]$. We shall take $\dot{u}\left(\cdot;q\right)$ to be the $H$-continuous second component of $x(\cdot;q)$.

At this point some remarks regarding the closure of the set $\overline{\mathrm{Dom}(A(q))}$ are in order. We show that under a slightly more restrictive boundedness condition on the operators $\mathcal{B}(q)$ than condition (B4), the operator $A(q)$ is in fact densely defined; that is, that $\overline{\mathrm{Dom}(A(q))} = \mathcal{H}(q)$.

<u>Theorem 3.2</u>  If condition $(B4)$ is replaced by the stronger condition that the operators $\mathcal{B}(q)$ map $H$-bounded subsets of $\mathrm{Dom}(\mathcal{B})$ into subsets of $V^*$ which are uniformly $V^*$-bounded in $q$ for $q \in Q$, then $\overline{\mathrm{Dom}(A(q))} = \mathcal{H}(q)$.

<u>Proof</u>:    Let $(\varphi,\psi) \in \mathcal{H}\left(q\right)$ and set

$$(\varphi_n,\psi_n) = J\left(\tfrac{1}{n};A\left(q\right)+\omega I\right)(\varphi,\psi) \in \mathrm{Dom}\left(A\left(q\right)\right),$$

for $n = 1,2,\ldots.$ As in the proof of Theorem 3.1, we find

$$(3.10) \qquad\qquad \varphi_n = \left(1+n^{-1}\omega\right)^{-1}\left(n^{-1}\psi_n + \varphi\right)$$

$$(3.11) \qquad\quad \left(T_{n^{-1}}\left(q\right)+n^{-1}\mathcal{B}\left(q\right)\right)\psi_n \ni \psi - n^{-1}\left(1+n^{-1}\omega\right)^{-1}\mathcal{A}\left(q\right)\varphi$$

where the operator $T_{n^{-1}}\left(q\right) \in \mathcal{L}\left(V,V^*\right)$ is given by (3.8) with $\lambda = n^{-1}$. Recalling (3.9), we have for $u \in V$

$$< T_{n^{-1}}\left(q\right)u,u > \ \geq\ \tau_1\left(n^{-1}\right)|u|^2 + \tau_0\left(n^{-1}\right)\|u\|^2,$$

where $\tau_1\left(n^{-1}\right) = \left(1+n^{-1}\omega\right) - n^{-2}\left(1+n^{-1}\omega\right)^{-1}\alpha_1 > 0$ for $n$ sufficiently large, and $\tau_0\left(n^{-1}\right) = n^{-2}\left(1+n^{-1}\omega\right)^{-1}\alpha_0$. Note that $\lim\limits_{n\to\infty}\tau_1\left(n^{-1}\right) = 1$ and $\lim\limits_{n\to\infty}\tau_0\left(n^{-1}\right) = 0$. Now from condition (B3) we find that for some $\tilde{\psi}_n \in \mathcal{B}\left(q\right)\psi_n$ and any $\tilde{\theta} \in \mathcal{B}\left(q\right)(0)$

(recall that $0 \in \text{Dom}(\mathcal{B})$ by condition (B1)),

$$\tau_1\left(n^{-1}\right)|\psi_n|^2 + \tau_0\left(n^{-1}\right)\|\psi_n\|^2 \leq\ <T_{n^{-1}}(q)\psi_n,\psi_n>$$

$$\leq\ <T_{n^{-1}}(q)\psi_n,\psi_n> +n^{-1}<\tilde{\psi}_n-\tilde{\theta},\psi_n>$$

$$=<\psi-n^{-1}\left(1+n^{-1}\omega\right)^{-1}\mathcal{A}(q)\varphi-n^{-1}\tilde{\theta},\psi_n>$$

$$\leq |\psi|\,|\psi_n| + n^{-1}\left(1+n^{-1}\omega\right)^{-1}\|\mathcal{A}(q)\varphi\|_*\|\psi_n\| + n^{-1}\|\tilde{\theta}\|_*\|\psi_n\|$$

$$\leq \frac{1}{2\tau_1(n^{-1})}|\psi|^2 + \frac{\tau_1(n^{-1})}{2}|\psi_n|^2 + \frac{(1+n^{-1}\omega)^{-1}}{\alpha_0}\|\mathcal{A}(q)\varphi\|_*^2$$

$$+ \frac{\tau_0(n^{-1})}{4}\|\psi_n\|^2 + \frac{(1+n^{-1}\omega)}{\alpha_0}\|\tilde{\theta}\|_*^2 + \frac{\tau_0(n^{-1})}{4}\|\psi_n\|^2.$$

Therefore

$$\frac{\tau_1(n^{-1})}{2}|\psi_n|^2 + \frac{\tau_0(n^{-1})}{2}\|\psi_n\|^2 \leq \frac{1}{2\tau_1(n^{-1})}|\psi|^2 + \frac{1}{\alpha_0}\|\mathcal{A}(q)\varphi\|_*^2 + \frac{(1+\omega)}{\alpha_0}\|\tilde{\theta}\|_*^2.$$

It follows that $\{\psi_n\}$ is uniformly bounded in $H$ and that $\left\{n^{-1}\|\psi_n\|\right\}$ is also uniformly bounded. From (3.11) we obtain

$$\psi_n-\psi \in -n^{-1}\omega\psi_n - n^{-2}\left(1+n^{-1}\omega\right)^{-1}\mathcal{A}(q)\psi_n - n^{-1}\left(1+n^{-1}\omega\right)^{-1}\mathcal{A}(q)\varphi - n^{-1}\mathcal{B}(q)\psi_n.$$

Therefore for any $\tilde{\psi}_n \in \mathcal{B}(q)\psi_n$, we have

$$\|\psi_n-\psi\|_* \leq n^{-1}\omega\|\psi_n\|_* + n^{-2}\left(1+n^{-1}\omega\right)^{-1}\|\mathcal{A}(q)\psi_n\|_*$$

$$+ n^{-1}\left(1+n^{-1}\omega\right)^{-1}\|\mathcal{A}(q)\varphi\|_* + n^{-1}\|\tilde{\psi}_n\|_*$$

$$\leq n^{-1}\omega\mu|\psi_n| + n^{-2}\alpha\|\psi_n\| + n^{-1}\alpha\|\varphi\| + n^{-1}\|\tilde{\psi}_n\|_*\ .$$

This estimate together with the uniform boundedness of $|\psi_n|$ and $n^{-1}\|\psi_n\|$ and the hypothesis of the theorem imply that $\lim_{n\to\infty}\psi_n = \psi$ in $V^*$. Thus using the uniform boundedness of $\{|\psi_n|\}$ and the density of $V$ in $H$, we obtain $\psi_n \to \psi$ weakly in $H$ as $n \to \infty$.

From (3.10) we find that

$$|\varphi_n| \leq n^{-1}|\psi_n| + |\varphi|$$

and

$$\|\varphi_n\| \leq n^{-1}\|\psi_n\| + \|\varphi\|$$

13

and therefore that the $\{\varphi_n\}$ are uniformly bounded in both $H$ and $V$. Since $(1 + n^{-1}\omega)\,\varphi_n$ $= n^{-1}\psi_n + \varphi$, we have

$$|\varphi_n - \varphi| \leq n^{-1}\,|\psi_n| + n^{-1}\omega\,|\varphi_n|$$

$$\leq (n^{-1} + n^{-2}\omega)\,|\psi_n| + n^{-1}\omega\,|\varphi| \to 0$$

as $n \to \infty$. Thus $\lim_{n\to\infty} \varphi_n = \varphi$ in $H$. Using the density of $H$ in $V^*$ and the uniform $V$-boundedness of the $\psi_n$, it is not difficult to argue that $\varphi_n \to \varphi$, weakly in $V$. Therefore $(\varphi_n, \psi_n) \to (\varphi, \psi)$ weakly in $X$ and thus the weak closure of Dom $(A(q))$ is all of $X$. Since $A(q) + \omega I$ is m-accretive by Theorem 3.1, the strong closure of Dom $(A(q))$ is convex. It follows, therefore, that $\overline{\mathrm{Dom}(A(q))} = \mathcal{H}(q)$ and the theorem is proved.

When we consider examples in Section 5 below, we shall show that there is an important class of nonlinear models for dissipation which satisfy the boundedness condition hypothesized in the statement of Theorem 3.2.

Finally, we note that another class of operators $\mathcal{B}(q)$ which lead to the operator $A(q)$ being densely defined is the one which includes operators $\mathcal{B}(q) \in \mathcal{L}(V, V^*)$ for which the set $\{\varphi \in V : \mathcal{B}(q)\varphi \in H\}$ is dense in $H$. For then condition (A3) implies that $\{\varphi \in V : \mathcal{A}(q)\varphi \in H\}$ is dense in $V$ (see [22], Theorem III.2.B) and consequently $\mathrm{Dom}(A(q)) = \{(\varphi, \psi) \in V \times V : \mathcal{A}(q)\,\varphi + \mathcal{B}(q)\,\psi \in H\}$ is dense in $\mathcal{H}(q)$. In particular, this will be the case for the well known linear Kelvin-Voigt (i.e., $\mathcal{B}(q) \sim \mathcal{A}(q)$) and the so-called structural (i.e. $\mathcal{B}(q) \sim \mathcal{A}(q)^{\frac{1}{2}}$) viscoelastic damping models.

## 4. Approximation Theory and Convergence Results

We apply Ritz-Galerkin finite element techniques to discretize the state equation. For each $n = 1, 2, \ldots$ let $H_n$ be a finite dimensional subspace of $H$ with $H_n \subset V$. Let $P_n : H \to H_n$ denote the orthogonal projection of $H$ onto $H_n$ computed with respect to the $H$ inner product. We assume that $P_n\mathrm{Dom}(\mathcal{B}) \subset \mathrm{Dom}(\mathcal{B})$ and that the approximation condition

$$\text{(P)} \qquad\qquad \lim_{n\to\infty} \|P_n\varphi - \varphi\| = 0, \qquad \varphi \in V,$$

14

is satisfied. We note that condition $(P)$ implies that $\lim\limits_{n\to\infty} |P_n\varphi - \varphi| = 0, \varphi \in H$ as well.

For each $n = 1, 2, \ldots$ and $q \in Q$, let $\mathcal{A}_n(q) \in \mathcal{L}(H_n) = \mathcal{L}(H_n, H_n)$ and $\mathcal{B}_n(q) :$Dom $(\mathcal{B}_n) \subset H_n \to 2^{H_n}$ denote the usual Galerkin approximations to the operators $\mathcal{A}(q)$ and $\mathcal{B}(q)$ respectively. More precisely, for $\varphi_n \in H_n$ let $\mathcal{A}_n(q)\varphi_n = \psi_n$ where $\psi_n$ is the unique element in $H_n$ guaranteed to exist by the Riesz representation theorem satisfying $< \mathcal{A}(q)\varphi_n, \chi_n > = < \psi_n, \chi_n >, \chi_n \in H_n$. Similarly, for $\varphi_n \in$Dom$(\mathcal{B}_n) \equiv$ Dom$(\mathcal{B}) \cap H_n \neq \emptyset$, let $\mathcal{B}_n(q)\varphi_n = \{\psi_n \in H_n :< \psi, \chi_n >=< \psi_n, \chi_n >, \chi_n \in H_n,$ for some $\psi \in \mathcal{B}(q)\varphi_n\}$. We set $f_n(\cdot; q) = P_n f(\cdot; q) \in L_1(0, T; H_n)$, $u_n^0(q) = P_n u^0(q)$ and $u_n^1(q) = P_n u^1(q)$.

Our approximation framework is based upon the following sequence of approximating identification problems.

$(\mathrm{ID}_n)$ For the observations $z \in Z$ given in problem $(ID)$, determine parameters $\bar{q}_n \in Q$ which minimize the functional

$$\Phi_n(q) = \phi(u_n(q), \dot{u}_n(q); z)$$

where for each $q \in Q$ $u_n(q) = u_n(\cdot; q)$ is the mild solution to the second order initial value problem in $H_n$ given by

$$(4.1) \qquad \ddot{u}_n(t) + \mathcal{B}_n(q)\dot{u}_n(t) + \mathcal{A}_n(q)u_n(t) \ni f_n(t; q), \quad 0 < t \leq T,$$

$$(4.2) \qquad u_n(0) = u_n^0(q), \qquad \dot{u}_n(0) = u_n^1(q).$$

Once again to establish the existence of a unique mild solution for each $q \in Q$, and to develop a convergence theory, we reformulate the initial value problem (4.1), (4.2) as an equivalent first order vector system in an appropriate product space, and then apply the abstract existence and approximation results outlined in Section 2.

For each $n = 1, 2, \ldots$ let $X_n = H_n \times H_n$ and for each $q \in Q$ let $\mathcal{H}_n(q)$ denote the space $X_n$ considered as a subspace of the Hilbert space $\mathcal{H}(q)$. Define the operator $A_n(q) :$ Dom

15

$(\mathcal{A}_n(q)) \subset \mathcal{H}_n(q) \to 2^{\mathcal{H}_n(q)}$ by

$$\mathcal{A}_n(q)(\varphi_n, \psi_n) = (-\psi_n, \mathcal{A}_n(q)\varphi_n + \mathcal{B}_n(q)\psi_n)$$

for each $(\varphi_n, \psi_n) \in \text{Dom}(\mathcal{A}_n(q)) \equiv H_n \times \text{Dom}(\mathcal{B}_n)$. Set $F_n(t; q) = (0, f_n(t; q))$ for almost every $t \in (0, T)$ and let $x_n^0(q) = (u_n^0(q), u_n^1(q))$. Note that $F_n(\cdot; q) \in L_1(0, T; \mathcal{H}_n(q))$ and that the assumptions that $x^0(q) \in \overline{\text{Dom}(A(q))}$ and $P_n \text{Dom}(\mathcal{B}) \subset \text{Dom}(\mathcal{B})$ imply that $x_n^0(q) \in \overline{\text{Dom}(\mathcal{A}(q))}$.

With the above definitions of $\mathcal{A}_n(q)$ and $\mathcal{B}_n(q)$, it is not difficult to argue that $\mathcal{A}_n(q) + \omega I$ is m-accretive in $\mathcal{H}_n(q)$. (The proof is analogous to the proof of Theorem 3.1.) It follows that for each $n = 1, 2, \ldots$ and each $q \in Q$, $\mathcal{A}_n(q)$ and $F_n(\cdot; q)$ generate a nonlinear evolution system, $\{U_n(t, s; q) : 0 \leq s \leq t \leq T\}$ on $\overline{\text{Dom}(\mathcal{A}(q))}$. By a mild solution $u_n(q) = u_n(\cdot; q)$ to the initial value problem (4.1), (4.2) we shall mean the $V$-continuous first component of $x_n(\cdot; q) = U_n(\cdot, 0; q) x_n^0(q)$. The $H$-continuous second component of $x_n(\cdot; q)$ will be taken to be $\dot{u}_n(\cdot; q)$.

The primary result of this paper is given in the following theorem.

<u>Theorem 4.1.</u>    For each $n = 1, 2, \ldots$ the problem $(ID_n)$ admits a solution $\bar{q}_n \in Q$. Moreover, the sequence $\{\bar{q}_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{\bar{q}_{n_k}\}_{n=1}^{\infty}$ with $\lim_{k \to \infty} \bar{q}_{n_k} = \bar{q} \in Q$, where $\bar{q}$ is a solution to problem $(ID)$.

<u>Proof:</u>    In light of the compactness of $Q$ and the continuity of $\phi$, the existence of a solution $\bar{q}_n \in Q$ to problem $(ID_n)$ can be established by demonstrating that the mapping $q \to x_n(\cdot \; ; q)$ is continuous from $Q \subset \mathcal{Q}$ into $C(0, T; X)$. Furthermore, as we have demonstrated in several of our earlier papers (see, for example, [6], [8]), showing that $\bar{q} = \lim_{k \to \infty} \bar{q}_{n_k}$ is indeed a solution to problem $(ID)$ requires arguing that for any convergent sequence $\{q_n\}_{n=1}^{\infty}$ with $q_n \in Q$ and $\lim_{n \to \infty} q_n = q_0 \in Q$ we have

(4.3)
$$\lim_{n \to \infty} |x_n(t; q_n) - x(t; q_0)|_X = 0$$

16

uniformly in $t$ for $t \in [0, T]$. Both the continuous dependence and the convergence (4.3) can be demonstrated via an application of Corollary 2.1. We shall establish (4.3) here; continuous dependence can be argued in an analogous manner.

Let $\{q_n\}_{n=1}^{\infty} \subset Q$ with $\lim_{n \to \infty} q_n = q_0$ in $Q$. The compactness of $Q$ of course implies that $q_0 \in Q$. In applying Theorem 2.1, we take $X_0 = \mathcal{H}(q_0), X_n = \mathcal{H}_n(q_n), |\cdot|_n = |\cdot|_{q_n}, n = 0, 1, 2, \dots$, and define $\pi_n \in \mathcal{L}(\mathcal{H}(q_0), \mathcal{H}_n(q_n))$ by $\pi_n(\varphi, \psi) = (P_n \varphi, P_n \psi), (\varphi, \psi) \in \mathcal{H}(q_0)$. Note that condition $(P)$ implies that the $\pi_n$ are uniformly bounded and converge strongly in $X$ to the identity. We set $F_n = F_n(\cdot; q_n), F_0 = F(\cdot; q_0), A_n = A_n(q_n), A_0 = A(q_0)$, and will use Corollary 2.1 to establish that

$$\lim_{n \to \infty} \left| U_n(t, 0; q_n) x_n^0(q_n) - U(t, 0; q_0) x^0(q_0) \right|_X = 0$$

uniformly in $t$, for $t \in [0, T]$. Under the general assumptions that we have made above, all of the hypotheses of Corollary 2.1 can be verified immediately with the exception of the resolvent convergence condition (ii). We establish it here.

Let $\lambda > 0$ be fixed, let $(\varphi, \psi) \in \mathcal{H}(q_0)$, $(\varphi_n, \psi_n) \in \mathcal{H}_n(q_n)$, and assume that $\lim_{n \to \infty} |(\varphi_n, \psi_n) - (\varphi, \psi)|_X = 0$. It then follows that $\lim_{n \to \infty} \varphi_n = \varphi$ in $V$ and $\lim_{n \to \infty} \psi_n = \psi$ in $H$. Setting $(\eta, \theta) = J(\lambda; A(q_0) + \omega I)(\phi, \psi) \in \text{Dom}(A(q_0))$ and $(\eta_n, \theta_n) = J(\lambda; A_n(q_n) + \omega I)(\varphi_n, \psi_n) \in \text{Dom}(A_n(q_n))$ we obtain

$$(4.4) \qquad \eta = (1 + \lambda \omega)^{-1} (\lambda \theta + \varphi)$$

$$(4.5) \qquad (T_\lambda(q_0) + \lambda B(q_0)) \theta \ni \psi - \lambda (1 + \lambda \omega)^{-1} A(q_0) \varphi$$

$$(4.6) \qquad \eta_n = (1 + \lambda \omega)^{-1} (\lambda \theta_n + \varphi_n)$$

$$(4.7) \qquad (T_{n,\lambda}(q_n) + \lambda B_n(q_n)) \theta_n \ni \psi_n - \lambda (1 + \lambda \omega)^{-1} A_n(q_n) \varphi_n,$$

where $T_\lambda(q)$ is given by (3.8), and

$$T_{n,\lambda}(q) = (1 + \lambda \omega) I + \lambda^2 (1 + \lambda \omega)^{-1} A_n(q) \in \mathcal{L}(H_n).$$

17

Recalling (3.9), we choose $\lambda > 0$ so that $\tau_1(\lambda) > 0$. We first show that the $\theta_n \in H_n$ are uniformly $V$-bounded in $n$. From (3.9), conditions (A4), (B3), and (B4), and the definitions of the operators $\mathcal{A}_n(q)$ and $\mathcal{B}_n(q)$ we find for some $\tilde{\theta}_n \in \mathcal{B}_n(q_n)\theta_n$ and any $\tilde{\xi}_n \in \mathcal{B}(q_n)(0)$

$$
\tau_1(\lambda)|\theta_n|^2 + \tau_0(\lambda)\|\theta_n\|^2 \leq\ <T_\lambda(q_n)\theta_n, \theta_n>
$$
$$
=\ <T_{n,\lambda}(q_n)\theta_n, \theta_n>
$$
$$
\leq\ <T_{n,\lambda}(q_n)\theta_n, \theta_n> + \lambda <\tilde{\theta}_n - \tilde{\xi}_n, \theta_n>
$$
$$
=\ <\psi_n - \lambda(1+\lambda\omega)^{-1}\mathcal{A}(q_n)\varphi_n - \lambda\tilde{\xi}_n, \theta_n>
$$
$$
\leq |\psi_n||\theta_n| + \lambda(1+\lambda\omega)^{-1}\alpha\|\varphi_n\|\|\theta_n\| + \lambda\beta_0\|\theta_n\|
$$
$$
\leq \frac{1}{2\tau_1(\lambda)}|\psi_n|^2 + \frac{\tau_1(\lambda)}{2}|\theta_n|^2 + \frac{\alpha^2(1+\lambda\omega)^{-1}}{\alpha_0}\|\theta_n\|^2
$$
$$
+ \frac{\tau_0(\lambda)}{4}\|\theta_n\|^2 + \frac{(1+\lambda\omega)}{\alpha_0}\beta_0^2 + \frac{\tau_0(\lambda)}{4}\|\theta_n\|^2
$$

where $\beta_0 = \sup\left\{\|\tilde{\xi}\|_* : \tilde{\xi} \in B(q)(0), q \in Q\right\}$. It follows that

$$
\frac{\tau_1(\lambda)}{2}|\theta_n|^2 + \frac{\tau_0(\lambda)}{2}\|\theta_n\|^2 \leq \frac{1}{2\tau_1(\lambda)}|\psi_n|^2 + \frac{\alpha^2(1+\lambda\omega)^{-1}}{\alpha_0}\|\varphi_n\|^2 + \frac{(1+\lambda\omega)}{\alpha_0}\beta_0^2.
$$

The fact that $\lim_{n\to\infty}\|\varphi_n - \varphi\| = 0$ and $\lim_{n\to\infty}|\psi_n - \psi| = 0$ implies that $\|\varphi_n\|$ and $|\psi_n|$ are uniformly bounded, and consequently that the $\theta_n$ are uniformly $V$-bounded.

Once again from (3.9), for any $\tilde{\theta}_n(q_n) \in \mathcal{B}(q_n)\theta_n$, $\tilde{\theta}(q_n) \in \mathcal{B}(q_n)\theta$, and $\tilde{\theta}(q_0) \in \mathcal{B}(q_0)\theta$ we have

$$
\tau_1(\lambda)|\theta_n - \theta|^2 + \tau_0(\lambda)\|\theta_n - \theta\|^2 \leq\ <T_\lambda(q_n)\{\theta_n - \theta\}, \theta_n - \theta>
$$
$$
\leq\ <T_\lambda(q_n)\{\theta_n - \theta\}, \theta_n - \theta> + \lambda <\tilde{\theta}_n(q_n) - \dot{\theta}(q_n), \theta_n - \theta>
$$
$$
=\ <T_{n,\lambda}(q_n)\theta_n - T_\lambda(q_0)\theta, \theta_n - P_n\theta> + \lambda <\tilde{\theta}_n(q_n) - \tilde{\theta}(q_0), \theta_n - P_n\theta>
$$
$$
+ <T_\lambda(q_n)\theta_n - T_\lambda(q_0)\theta, P_n\theta - \theta> + \lambda <\tilde{\theta}_n(q_n) - \tilde{\theta}(q_0), P_n\theta - \theta>
$$
$$
+ <T_\lambda(q_0)\theta - T_\lambda(q_n)\theta, \theta_n - \theta> + \lambda <\tilde{\theta}(q_0) - \tilde{\theta}(q_n), \theta_n - \theta>.
$$

Now from (4.7) there exists a $\tilde{\theta}_n \in \mathcal{B}_n(q_n)\theta_n$ for which

$$
T_{n,\lambda}(q_n)\theta_n + \lambda\tilde{\theta}_n = \psi_n - \lambda(1+\lambda\omega)^{-1}\mathcal{A}_n(q_n)\varphi_n.
$$

Choose $\tilde{\theta}_n(q_n)$ to be any element in $\mathcal{B}(q_n)\theta_n$ for which

$$< \tilde{\theta}_n(q_n), \chi_n > \; = \; < \tilde{\theta}_n, \chi_n >, \qquad \chi_n \in H_n.$$

That such a $\tilde{\theta}_n(q_n) \in \mathcal{B}(q_n)\theta_n$ exists is "guaranteed" by the definition of $\mathcal{B}_n(q_n)$. Also, recalling (4.5), we choose $\tilde{\theta}(q_0) \in \mathcal{B}(q_0)\theta$ so that $T_\lambda(q_0)\theta + \lambda\tilde{\theta}(q_0) = \psi - \lambda(1 + \lambda\omega)^{-1} \mathcal{A}(q_0)\varphi$. Then

$$\tau_1(\lambda) |\theta_n - \theta|^2 + \tau_0(\lambda) \|\theta_n - \theta\|^2$$

$$\leq \; < T_{n,\lambda}(q_n)\theta_n + \lambda\tilde{\theta}_n - \{T_\lambda(q_0)\theta + \lambda\tilde{\theta}(q_0)\}, \theta_n - P_n\theta >$$

$$+ \; < T_\lambda(q_n)\theta_n - T_\lambda(q_0)\theta, P_n\theta - \theta > + \lambda < \tilde{\theta}_n(q_n) - \tilde{\theta}(q_0), P_n\theta - \theta >$$

$$+ < T_\lambda(q_0)\theta - T_\lambda(q_n)\theta, \theta_n - \theta > + \lambda < \tilde{\theta}(q_0) - \tilde{\theta}(q_n), \theta_n - \theta >$$

$$= \; < \psi_n - \psi, \theta_n - P_n\theta > + \lambda(1 + \lambda\omega)^{-1} < \mathcal{A}(q_0)\varphi - \mathcal{A}_n(q_n)\varphi_n, \theta_n - P_n\theta >$$

$$+ < T_\lambda(q_n)\theta_n - T_\lambda(q_0)\theta, P_n\theta - \theta > + \lambda < \tilde{\theta}_n(q_n) - \tilde{\theta}(q_0), P_n\theta - \theta >$$

$$+ < T_\lambda(q_0)\theta - T_\lambda(q_n)\theta, \theta_n - \theta > + \lambda < \tilde{\theta}(q_0) - \tilde{\theta}(q_n), \theta_n - \theta >$$

$$= \; < \psi_n - \psi, \theta_n - P_n\theta > + \lambda(1 + \lambda\omega)^{-1} < \mathcal{A}(q_0)\varphi - \mathcal{A}(q_n)\varphi, \theta_n - P_n\theta >$$

$$+ \lambda(1 + \lambda\omega)^{-1} < \mathcal{A}(q_n)(\varphi_n - \varphi), \theta_n - P_n\theta > + (1 + \lambda\omega) < \theta_n - \theta, P_n\theta - \theta >$$

$$+ \lambda^2(1 + \lambda\omega)^{-1} < \mathcal{A}(q_n)\theta_n - \mathcal{A}(q_0)\theta, P_n\theta - \theta > + \lambda < \tilde{\theta}_n(q_n) - \tilde{\theta}(q_0), P_n\theta - \theta >$$

$$+ \lambda^2(1 + \lambda\omega)^{-1} < (\mathcal{A}(q_0)\theta - \mathcal{A}(q_n)\theta, \theta_n - \theta > + \lambda < \tilde{\theta}(q_0) - \tilde{\theta}(q_n), \theta_n - \theta >$$

$$\leq |\psi_n - \psi| \, |\theta_n - P_n\theta| + \lambda(1 + \lambda\omega)^{-1} \|\mathcal{A}(q_0)\varphi - \mathcal{A}(q_n)\varphi\|_* \|\theta_n - P_n\theta\|$$

$$+ \lambda(1 + \lambda\omega)^{-1} \|\mathcal{A}(q_n)(\varphi_n - \varphi)\|_* \|\theta_n - P_n\theta\| + (1 + \lambda\omega) |\theta_n - \theta| \, |P_n\theta - \theta|$$

$$+ \lambda^2(1 + \lambda\omega)^{-1} \|\mathcal{A}(q_n)\theta_n - \mathcal{A}(q_0)\theta\|_* \|P_n\theta - \theta\| + \lambda\|\tilde{\theta}_n(q_n) - \tilde{\theta}(q_0)\|_* \|P_n\theta - \theta\|$$

$$+ \lambda^2(1 + \lambda\omega)^{-1} \|\mathcal{A}(q_0)\theta - \mathcal{A}(q_n)\theta\|_* \|\theta_n - \theta\| + \lambda\|\tilde{\theta}(q_0) - \tilde{\theta}(q_n)\|_* \|\theta_n - \theta\|$$

$$\leq \{|\theta_n| + |\theta|\} |\psi_n - \psi| + \lambda(1 + \lambda\omega)^{-1} \{\|\theta_n\| + K\|\theta\|\} \|\mathcal{A}(q_0)\varphi - \mathcal{A}(q_n)\varphi\|_*$$

$$+ \lambda(1 + \lambda\omega)^{-1} \alpha \{\|\theta_n\| + K\|\theta\|\} \|\varphi_n - \varphi\| + (1 + \lambda\omega)\{|\theta_n| + |\theta|\} |P_n\theta - \theta|$$

$$+ \lambda^2 (1 + \lambda\omega)^{-1} \alpha \{\|\theta_n\| + \|\theta\|\} \|P_n\theta - \theta\| + \lambda\{\|\tilde{\theta}_n(q_n)\|_* + \|\tilde{\theta}(q_0)\|_*\} \|P_n\theta - \theta\|$$

$$+ \lambda^2 (1 + \lambda\omega)^{-1} \{\|\theta_n\| + \|\theta\|\} \|\mathcal{A}(q_0)\theta - \mathcal{A}(q_n)\theta\|_* + \lambda\{\|\theta_n\| + \|\theta\|\} \|\tilde{\theta}(q_0) - \tilde{\theta}(q_n)\|_*$$

where $K$ is the uniform bound on the operators $P_n \in \mathcal{L}(V, V)$ guaranteed to exist by condition $(P)$. By condition $(B2)$ we can choose $\tilde{\theta}(q_n) \in \mathcal{B}(q_n)\theta$ so that $\lim_{n\to\infty} \tilde{\theta}(q_n) = \tilde{\theta}(q_0)$ in $V^*$. Thus, the fact that $\lim_{n\to\infty} \psi_n = \psi$ in $H$, $\lim_{n\to\infty} \varphi_n = \varphi$ in $V$, the uniform boundedness of $\|\theta_n\|$ (and therefore $|\theta_n|$ as well), the continuity condition $(A2)$, and the boundedness condition $(B4)$ imply that $\lim_{n\to\infty} \theta_n = \theta$ in both $H$ and $V$. From (4.4) and (4.6) it then immediately follows that $\lim_{n\to\infty} \eta_n = \eta$ in $V$ and the theorem is proved.

Remark   In practice it is frequently the case that the admissible parameter set $Q$ is also infinite dimensional; for example, when the unknown parameters to be identified are elements in a function space. In this situation the set $Q$ must also be discretized and Theorem 4.1 must, to a certain degree, be modified. For each m = 1, 2, ... let $I^m : Q \subset Q \to Q$ be a continuous map with range $Q^m = I^m(Q)$ in a finite dimensional subspace of $Q$ and with the property that $\lim_{m\to\infty} I^m(q) = q$, uniformly in $q$ for $q \in Q$. We then define the doubly indexed sequence of approximating identification problems $(ID_n^m)$ by letting problem $(ID_n^m)$ be the problem $(ID_n)$ with the set $Q$ replaced by the set $Q^m$. The modification to Theorem 4.1 would state that each of these problems admits a solution $\bar{q}_n^m \in Q^m$ and that the sequence $\{\bar{q}_n^m\}$ will have a $Q$-convergent subsequence whose limit is in $Q$ and is a solution to problem $(ID)$. We note that each of the problems $(ID_n^m)$ involves the minimization of a continuous functional over a compact subset of Euclidean space subject to finite dimensional state space (ODE) constraints. As such, for each m and n, problem $(ID_n^m)$ may be solved using standard computational algorithms and techniques.

## 5. An Example

In order to illustrate the application of the general theory that was presented above, we consider the problem of estimating or identifying the nonlinear damping term in the forced one dimensional wave equation

$$(5.1) \qquad \frac{\partial^2 u}{\partial t^2}(t,\eta) + b(q)\left(\frac{\partial u}{\partial t}(t,\eta)\right) - a\frac{\partial^2 u}{\partial \eta^2}(t,\eta) \ni f(t,\eta), t > 0, \quad 0 < \eta < 1$$

with the Dirichlet boundary conditions

$$(5.2) \qquad u(t,0) = 0, \qquad u(t,1) = 0, \quad t > 0$$

and initial data

$$(5.3) \qquad u(0,\eta) = u^0(\eta), \ \frac{\partial u}{\partial t}(0,\eta) = u^1(\eta), \ 0 < \eta < 1.$$

We note that for definiteness we have chosen the Dirichlet boundary conditions (5.2), however all of the discussion below would remain with any of the usual self-adjoint boundary conditions (i.e. Neumann, Robin, etc.). We assume that $a > 0$, $f \in L_2((0,T) \times (0,1))$, $u^0 \in H_0^1(0,1)$ and $u^1 \in L_2(0,1)$. Let $\mathcal{Q}$ be a metric space and let $Q \subset \mathcal{Q}$ be compact. For each $q \in Q$ we assume that the mapping $b(q)(\cdot) : \mathbf{R} \to 2^{\mathbf{R}}$ satisfies the following conditions:

(b1) The set $\{\varphi \in H_0^1(0,1)$: there exists a $\psi \in H^{-1}(0,1)$ such that $\psi(\eta) \in b(q)(\varphi(\eta))$, a.e. $\eta \in (0,1)\}$ is independent of $q \in Q$, and $0 \in b(q)(0)$,

(b2) The mapping $q \to b(q)(\theta)$ is lower semi-continuous from $Q \subset \mathcal{Q}$ into $2^{\mathbf{R}}$ for almost every $\theta \in \mathbf{R}$,

(b3) For each $q \in Q$ the mapping $b(q)(\cdot)$ is nondecreasing and for some $\lambda > 0$ the inclusion $\theta + \lambda b(q)(\theta) \ni \xi$ has a solution $\theta \in \mathbf{R}$ for each $\xi \in \mathbf{R}$ (i.e. $b(q)(\cdot)$ is maximal monotone in $\mathbf{R}$),

(b4) There exists a polynomial $p$, independent of $q \in Q$ for which $|\tilde{\theta}| \leq p(|\theta|)$ for all $\tilde{\theta} \in b(q)(|\theta|)$, and almost every $\theta \in \mathbf{R}$.

21

To reformulate the initial-boundary value problem (5.1) -- (5.3) as an abstract system of the form (3.1), (3.2), we set $H = L_2(0,1)$ endowed with the standard inner product, and let $V = H_0^1(0,1)$ be endowed with the norm $\|\varphi\| = \int_0^1 |D\varphi(\eta)|^2 \, d\eta$. In this case we have $V^* = H^{-1}(0,1)$ together with the dense and continuous embeddings $V \hookrightarrow H \hookrightarrow V^*$. We define the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ by

$$
(5.4) \qquad < \mathcal{A}\varphi, \psi > = a \int_0^1 D\varphi(\eta) \, D\psi(\eta) \, d\eta, \quad \varphi, \psi \in H_0^1(0,1).
$$

It is easily shown and well known that the operator $\mathcal{A}$ given by (5.4) satisfies conditions (A1) - (A4). (Note that in the present example the operator $\mathcal{A}$ is assumed known and as such does not depend on $q$.)

To define the operator $\mathcal{B}(q)$ we follow the treatment in [18] and use the notion of a subdifferential of a proper convex lower semicontinuous mapping (see [10]). For each q $\in Q$ let $b_0(q)(\cdot)$ denote the minimal section of the mapping $b(q)(\cdot)$. That is $b_0(q)(\cdot)$ is the single-valued mapping from $\mathbf{R}$ into $\mathbf{R}$ defined by $b_0(q) = \tilde{\theta}$, where $\tilde{\theta}$ is the unique element in $b(q)(\theta)$ of minimal absolute value. That the minimal section of $b(q)(\cdot)$ is well defined is a consequence of condition (b3). Since $\text{Dom}(b(q)(\cdot)) = \mathbf{R}$, the proper, convex, lower semi -continuous function $j(\cdot; q) : \mathbf{R} \to \bar{\mathbf{R}}$ can be defined by

$$
j(\theta; q) = \int_0^\theta b_0(q)(\xi) \, d\xi
$$

where $\bar{\mathbf{R}}$ denotes the extended real numbers. It can be shown that $j(\cdot; q)$ is bounded below by an affine function so that $j(\theta; q) > -\infty$, $\theta \in \mathbf{R}$, and that $\partial j(\cdot; q) = b(q)(\cdot)$, where $\partial$ denotes the subdifferential operator. For each $q \in Q$ define $\gamma(\cdot; q) : H_0^1(0,1) \to \bar{\mathbf{R}}$ by

$$
\gamma(\varphi; q) = \begin{cases} \int_0^1 j(\varphi(\eta); q) \, d\eta & \text{if } j(\varphi(\cdot); q) \in L_1(0,1) \\ +\infty & \text{otherwise.} \end{cases}
$$

Then $\gamma(\cdot; q)$ is also proper, convex and lower semi-continuous and we define $\mathcal{B}(q) :$ $\text{Dom}(\mathcal{B}) \subset V \to 2^{V^*}$ by

$$
(5.5) \qquad \mathcal{B}(q)\varphi = \partial \gamma(\varphi; q),
$$

where $\varphi \in \text{Dom}(\mathcal{B}(q)) = \{\varphi \in H_0^1(0,1) : \partial\gamma(\varphi;q) \neq \emptyset\}$. (The fact that $\text{Dom}(\mathcal{B}(q)) = \text{Dom}(\mathcal{B})$ is independent of $q$ is a consequence of condition (b1). ) It can be shown (see, for example, [10, p. 61]) that for $\varphi \in \text{Dom}(\mathcal{B}), \psi \in \mathcal{B}(q)\varphi$ if and only if $\psi(\eta) \in b(q)(\varphi(\eta))$, a.e. $\eta \in (0,1)$, and via conditions (b1)-(b4), that the operators $\mathcal{B}(q)$ given by (5.5) satisfy conditions (B1) - (B4).

To illustrate the formulation of an inverse problem we take the observation space $Z$ to be given by $Z = \underset{i=1}{\overset{\nu}{\times}} \{\mathbf{R}^\ell \times L_2(0,1)\}$ and the performance index from $C(0,T; V \times H)$ into the nonnegative reals to be given by the weighted least-squares functional

$$\phi(u, \dot{u}; z) = \sum_{i=1}^{\nu} \{\rho_i \sum_{j=1}^{\ell} |u(t_i, \eta_j) - z_{i,j}^1|^2 + \sigma_i \int_0^1 |\dot{u}(t_i, \eta) - z_i^2(\eta)|^2 d\eta\}$$

for $(u, \dot{u}) \in C(0,T; V \times H)$ and $z = ((z_1^1, z_1^2), \ldots (z_\nu^1, z_\nu^2)) \in Z$ with $\rho_i, \sigma_i \geq 0, i = 1, 2, \ldots, \nu$, $0 < t_1 < t_2 < \ldots < t_\nu \leq T$, and $0 < \eta_1 < \eta_2 \ldots < \eta_\ell < 1$.

We provide two specific examples of possible choices of the parameter space $\mathcal{Q}$, the admissible parameter set $Q$, and mappings $b(q)(\cdot) : \mathbf{R} \to 2^{\mathbf{R}}$ satifying conditions (b1) - (b4). A relatively simple example might involve the estimation of the constant parameters $q = (\alpha_0, \beta_0, \theta_0)$ in the saturating polynomial function

$$b(q)(\theta) = \begin{cases} \beta_0 |\theta|^{\alpha_0} sgn(\theta), & -\theta_0 \leq \theta \leq \theta_0 \\ \beta_0 |\theta_0|^{\alpha_0} sgn(\theta), & |\theta| > \theta_0. \end{cases}$$

In this case we would have $\mathcal{Q} = \mathbf{R}^3$ and $Q = \{(\alpha_0, \beta_0, \theta_0) : 0 \leq \alpha_0 \leq \tilde{\alpha}, 0 \leq \beta_0 \leq \tilde{\beta}, 0 \leq \theta_0 \leq \tilde{\theta}\}$ for some $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta} \geq 0$ given and fixed. We note that when $\alpha_0 = 0$, in order to satisfy the maximality in condition (b3) we must take

$$b(q)(\theta) = \begin{cases} \beta_0 sgn(\theta), & \theta \neq 0 \\ [-\beta_0, \beta_0] & \theta = 0. \end{cases}$$

We note also that when $\tilde{\alpha} \leq 2$ we have $\text{Dom}(\mathcal{B}) = V$ and that the operators $\mathcal{B}(q)$ given by (5.5) map $H$-bounded subsets of $V$ into $V^*$-bounded subsets, uniformly in $q$ for $q \in Q$. Thus the hypotheses of Theorem 3.2 are satisfied and we see that $\overline{\text{Dom}(A(q))} = V \times H$.

A second, somewhat more challenging problem might involve the estimation of a functional parameter. In this case we take $\mathcal{Q} = C_B(\mathbf{R})$, the set of all bounded, continuous, real valued functions on $\mathbf{R}$ endowed with the supremum norm. We take $Q$ to be the $\mathcal{Q}$-closure of the set

$$\{q \in Q : q(\theta) = -q(-\theta), \theta q(\theta) \geq 0, \text{for } \theta \in \mathbf{R},$$

(5.6)
$$|q(\theta)| = q(\theta_0), \text{for } |\theta| \geq \theta_0, q \in H^1(-\theta_0, \theta_0),$$

$$|q|_{H^1(-\theta_0, \theta_0)} \leq K_0, \quad q'(\theta) \geq 0, \text{ for a.e. } \theta \in (-\theta_0, \theta_0)\}$$

where $\theta_0$ and $K_0$ are given positive constants. It is not difficult to argue (via the Arzela-Ascoli Theorem) that $Q$ is a compact subset of $\mathcal{Q}$. For $q \in Q$ we set $b(q)(\theta) = q(\theta), \theta \in \mathbf{R}$.

With regard to illustrating the approximation or discretization of the state, we briefly outline a linear spline-based scheme. For each $n = 1, 2, \ldots$, let $H_n = span\{\varphi_n^j\}_{j=1}^{n-1}$ where $\phi_n^j$ is the $j - th$ linear $B$-spline on the interval $[0, 1]$ defined with respect to the uniform mesh $\{0, 1/n, 2/n, \ldots, 1\}$. That is,

$$\varphi_n^j(\eta) = \begin{cases} 0 & 0 \leq \eta \leq \frac{i-1}{n} \\ n\eta - j + 1 & \frac{i-1}{n} \leq \eta \leq \frac{i}{n} \\ j + 1 - n\eta & \frac{i}{n} \leq \eta \leq \frac{i+1}{n} \\ 0 & \frac{i+1}{n} \leq \eta \leq 1, \end{cases}$$

$j = 1, 2, \ldots, n - 1$. It is immediately clear that $H_n \subset V = H_0^1(0, 1)$, for each $n$. Let $P_n : H \to H_n$ denote the orthogonal projection of $L_2(0, 1)$ onto $H_n$ computed with respect to the usual $L_2$ inner product. Using well known estimates for interpolatory splines (for example, those found in [23]) it can be argued that condition $(P)$ is satisfied.

In the example involving the identification of a functional parameter, the admissible parameter set $Q$ chosen to be the $\mathcal{Q}$-closure of the set given in (5.6) can be discretized as follows. For each $m = 1, 2, \ldots$ and $\Theta \in \mathbf{R}^+$ let $\{\psi_j^m(\cdot, \Theta)\}_{j=0}^m$ denote the standard linear $B$-splines on the interval $[0, \Theta]$ defined with respect to the uniform mesh $\{0, \frac{\Theta}{m}, \frac{2\Theta}{m}, \ldots, \Theta\}$ and then extended to a continuous function on the entire positive real line via $\psi_j^m(\theta; \Theta) = \psi_j^m(\Theta; \Theta), \theta \geq \Theta$. For $q \in Q$ set

(5.7)
$$(I^m q)(\theta) = \sum_{j=1}^m q\left(\frac{j\theta_q}{m}\right) \psi_j^m(|\theta|; \theta_q) sgn(\theta)$$

for $\theta \in \mathbf{R}$ where $\theta_q$ is the number in $(0, \theta_0)$ for which $|q(\theta)| = q(\theta_q), |\theta| \geq \theta_q$. (Note that the lower limit in the sum in (5.7) is 1 rather than 0 since $q \in Q$ implies $q(0) = 0$.) Using the Peano Kernel Theorem (see [21] p. 22) it can be argued that

$$\|I^m q - q\|_\infty = \sup_{\mathbf{R}} |I^m q - q| \leq \frac{1}{2} \left(\frac{\theta_0}{m}\right)^{1/2} K_0$$

and consequently that $\lim_{m \to \infty} I^m q = q$, uniformly in $q$ for $q \in Q$. We would then set $Q^m = I^m(Q)$.

For $H_n$ and $Q^m$ as defined above, the finite dimensional initial value problem (4.1), (4.2) takes the form

$$(5.8) \qquad M_n \ddot{w}_n(t) + C_n(\dot{w}_n(t); q^m) + K_n w_n(t) = F_n(t), \ 0 < t \leq T,$$

$$(5.9) \qquad M_n w_n(0) = w_n^0, \ M_n \dot{w}_n(0) = w_n^1,$$

where $F_n(t), \omega_n^0$, and $w_n^1$ are the (n-1)-vectors whose i-th components are given by $F_n^i(t) = <f(t), \varphi_n^i>, \omega_n^{0i} = <u^0, \varphi_n^i>$ and $w_n^{1i} = <u^1, \varphi_n^i>$, respectively, and $M_n$ and $K_n$ are the $(n-1) \times (n-1)$ matrices whose (i,j)-th entries are given by $M_n^{i,j} = <\varphi_n^i, \varphi_n^j>$ and $K_n^{i,j} = <a\varphi_n^i, \varphi_n^j>$. The vector function $C_n(\cdot \ ; q^m) : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is given by

$$C_n^i(v; q^m) = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \{nx - i + 1\} q^m \left(\{nx - i\} \{v^i - v^{i-1}\} + v^i\right) dx$$

$$+ \int_{\frac{i}{n}}^{\frac{i+1}{n}} \{i + 1 - nx\} q^m \left(\{nx - i\} \{v^{i+1} - v^i\} + v^i\right) dx,$$

$i = 1, 2, \cdots, n - 1$, for $v \in \mathbf{R}^{n-1}$ with $v^0, v^n = 0$. If $w_n(\cdot \ ; q^m)$ is the solution to the second order initial value problem (5.8), (5.9) corresponding to $q^m \in Q^m$, then $u_n(t; q^m) = \sum_{j=1}^{n-1} w_n^j(t; q^m) \varphi_n^j$ and $\dot{u}_n(t; q^m) = \sum_{j=1}^{n-1} \dot{w}_n^j(t; q^m) \varphi_n^j$, for $t \in [0, T]$. If $q^m \in Q^m$ is given by $q^m(\theta) = \sum_{j=1}^{m} q_j^m \psi_j^m(|\theta|; \theta_{q^m}) \ sgn(\theta), \ \theta \in \mathbf{R}$, the identification problem $(ID_n^m)$

25

becomes one of determining parameters $(\bar{q}_1^m, ..., \bar{q}_m^m, \bar{\theta}_{q^m})$ in some compact subset of $\mathbf{R}^{m+1}$ which minimize $\Phi_n(q^m) = \phi(u_n(q^m), \dot{u}_n(q^m); z)$.

A discussion of some implementation questions relevant to the schemes outlined (in particular with regard to supercomputing) together with numerical results can be found in [6].

# References

[1] A.V. Balakrishnan, A theory of nonlinear damping in flexible structures, in <u>Stabilization of Flexible Structures</u>, (A.V. Balakrishnan J.P. Zolesio, eds.), Optimization Software, Inc., Los Angeles, 1988, pp. 1 - 12.

[2] H.T. Banks, R.H. Fabiano, Y. Wang, D.J. Inman and H.H. Cudney, Spatial versus time hysteresis in damping mechanisms, Proc. 27th IEEE Conf. on Dec. and Control (Austin, TX), 1988, pp. 1674 - 1677.

[3] H.T. Banks and D.J. Inman, On damping mechanisms in beams, ASME J. Applied Mechanics, submitted.

[4] H.T. Banks and K. Ito, A unified framework for approximation and inverse problems for distributed parameter systems, Control-Theory and Advanced Technology 4 (1988), 73 - 90.

[5] H.T. Banks and K. Kunisch, <u>Estimation Techniques for Distributed Parameter Systems</u>, Birkhäuser, Boston, 1989.

[6] H.T. Banks, C.K. Lo, S. Reich and I.G. Rosen, Numerical studies of identification in nonlinear distributed parameter systems, Proceedings of the Fourth International Conference on the Identification and Control of Distributed Parameter Systems, Vorau, Austria, July 10 - 16, 1988, Birkhäuser, to appear.

[7] H.T. Banks, R.K. Powers, and I.G. Rosen, Inverse problems in the modeling of vibrations of flexible beams, in <u>Distributed Parameter Systems</u> (F. Kappel, K. Kunisch, and W. Schappacher, eds.), Springer Lec. Notes in Control & Infor. Sci. <u>102</u> (1987), pp. 1 - 22.

[8] H.T. Banks, S. Reich and I.G. Rosen, An approximation theory for the identification

of nonlinear distribut.. parameter systems, ICASE Report No. 88-26, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, April, 1988, SIAM J. Control and Opt., to appear.

[9] H.T. Banks, Y. Wang, D.J. Inman and H.H. Cudney, Parameter indentification techniques for the estimation of damping in flexible structures experiments, Proc. 26th IEEE Conf. on Dec. and Control (Los Angeles, CA), 1987, pp. 1392 - 1395.

[10] V. Barbu, <u>Nonlinear Semigroups and Differential Equations in Banach Spaces</u>, Noordhoff International, Leyden, The Netherlands, 1976.

[11] R.W. Clough and J. Penzien, <u>Dynamics of Structures</u>, McGraw-Hill, New York, 1975.

[12] M.G. Crandall and L.C. Evans, On the relation of the operator $\partial/\partial s + \partial/\partial t$ to evolution governed by accretive operators, Israel J. Math. 21 (1975), 261 - 278.

[13] M.G. Crandall and A. Pazy, Nonlinear evolution equations in Banach Space, Israel J. Math. 11 (1972), 57 - 94.

[14] L.C. Evans, Nonlinear evolution equations in an arbitrary Banach space, Israel J. Math. 26 (1977), 1 - 42.

[15] J.A. Goldstein, Approximation of nonlinear semigroups and evolution equations, J. Math. Soc Japan 24 (1972), 558 - 573.

[16] S. G. Krein, <u>Linear Differential Equations in Banach Space</u>, American Mathematical Society Translations of Mathematical Monographs, Volume 29, American Mathematical Society, Providence, R.I., 1971.

[17] K. Kuratowski, <u>Topology</u>, Volume I, Academic Press, New York and PWN-Polish Sci-

entific Publishers, Warszawa, 1966.

[18] G. Morosanu, <u>Nonlinear Evolution Equations and Applications</u>, D. Reidel, Dordrecht, Holland, 1988.

[19] A.H. Nayfeh and D.T. Mook, <u>Nonlinear Oscillations</u>, John Wiley, New York, 1979.

[20] E.P. Popov, <u>Introduction to Mechanics of Solids</u>, Prentice Hall, Englewood Cliffs, N.J., 1968.

[21] M.H. Schultz, <u>Spline Analysis</u>, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[22] R.E. Showalter, <u>Hilbert Space Methods for Partial Differential Equations</u> Pitman, London, 1977.

[23] B.K. Swartz and R.S. Varga, Error bounds for spline and L-spline interpolation, J. Approx. Theory 6 (1972), 6 - 49.

[24] S. Timoshenko, D.H. Young and W. Weaver, <u>Vibration Problems in Engineering</u>, J. Wiley, New York, 1974.

| NASA National Aeronautics and Space Administration | Report Documentation Page | |
|---|---|---|

| 1. Report No. NASA CR-181890 ICASE Report No. 89-16 | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|

| 4. Title and Subtitle | | 5. Report Date |
|---|---|---|
| ESTIMATION ON NONLINEAR DAMPING IN SECOND ORDER DISTRIBUTED PARAMETER SYSTEMS | | August 1989 |
| | | 6. Performing Organization Code |

| 7. Author(s) H. T. Banks S. Reich I. G. Rosen | 8. Performing Organization Report No. 89-16 |
|---|---|
| | 10. Work Unit No. |

| 9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225 | 505-90-21-01 |
|---|---|
| | 11. Contract or Grant No. NAS1-18107, NAS1-18605 |
| | 13. Type of Report and Period Covered |

| 12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center. Hampton, VA 23665-5225 | Contractor Report |
|---|---|
| | 14. Sponsoring Agency Code |

15. Supplementary Notes

Langley Technical Monitor:          Control Theory and Advanced
Richard W. Barnwell                      Technology

Final Report

16. Abstract

An approximation and convergence theory for the identification of nonlinear damping in abstract wave equations is developed. It is assumed that the unknown dissipation mechanism to be identified can be described by a maximal monotone operator acting on the generalized velocity. The stiffness is assumed to be linear and symmetric. Functional analytic techniques are used to establish that solutions to a sequence of finite dimensional (Galerkin) approximating identification problems in some sense approximate a solution to the original infinite dimensional inverse problem.

| 17. Key Words (Suggested by Author(s)) accretive operator, Galerkin approxiamtion, inverse problems, nonlinear damping, nonlinear evolution equation. | 18. Distribution Statement 64 - Numerical Analysis Unclassified - Unlimited | |
|---|---|---|
| 19. Security Classif. (of this report) Unclassified | 20. Security Classif. (of this page) Unclassified | 21. No. of pages 31 | 22. Price A03 |

NASA FORM 1626 OCT 86